Exc 1 [2.0 pt]

- Create homogeneous Dirichlet condition

$$u = \bar{u} + \tilde{u} \quad \text{s.t} \quad \tilde{u}(1) = 0 \quad \Rightarrow \bar{u} = -2$$

$$\Rightarrow u = -2 + \tilde{u}$$

0.5 then $\frac{du}{dx}(0) - u(0) = 3 \Rightarrow \frac{d\tilde{u}}{dx}(0) - (-2 + \tilde{u}(0)) = 3$

$$\Rightarrow \frac{d\tilde{u}}{dx}(0) - \tilde{u}(0) = 1$$

and $-\frac{d}{dx}\left(\cos x \frac{d\tilde{u}}{dx}\right) + \frac{d\tilde{u}}{dx} = \tan x$

- weak form (we will drop $\sim$)

0.3 2 residuals $r_1(u) = \tan x - \left(-\frac{d}{dx}\left(\cos x \frac{du}{dx}\right)\right) + \frac{du}{dx}$

$$r_2(u) = 1 - \left(\frac{du}{dx}(0) - u(0)\right)$$

0.3 $\mathcal{V} = \{ v \in H^1(0,1) \mid v(1) = 0 \}$

find $u \in \mathcal{V}$ s.t $(v, r_1(u)) + \alpha\, v(0)\, r_2(u) = 0 \quad \forall v \in \mathcal{V}$

$$(v, r_1(u)) + \alpha v(0) r_2(u) = 0$$

0.3 $\Rightarrow \left(v, -\frac{d}{dx}\left(\cos x \frac{du}{dx}\right)\right) + \left(v, \frac{du}{dx}\right) + \alpha v(0)\left(\frac{du}{dx}(0) - u(0)\right)$

$$= (v, \tan x) + \alpha v(0)$$

$\underset{\text{p.i}}{\Rightarrow}$ $-v \cos x \frac{du}{dx}\Big|_0^1 + \left(\frac{dv}{dx}, \cos x \frac{du}{dx}\right) + \left(v, \frac{du}{dx}\right)$

$$+ \alpha\, v(0)\left(\frac{du}{dx}(0) - u(0)\right) = (v, \tan x) + \alpha v(0)$$

$\underset{v(1)=0}{\Rightarrow}$ $\left(\frac{dv}{dx}, \cos x \frac{du}{dx}\right) + \left(v, \frac{du}{dx}\right) + v(0) \cos 0 \frac{du}{dx}(0)$

0.6 $+ \alpha v(0) \frac{du}{dx}(0) - \alpha v(0) u(0) = (v, \tan x) + \alpha v(0)$

find $u \in \mathcal{V}$ s.t $a(v,u) = F(v) \quad \forall v \in \mathcal{V}$

with $a(v,u) = \left(\frac{dv}{dx}, \cos x \frac{du}{dx}\right) + \left(v, \frac{du}{dx}\right) + (1+\alpha)\, v(0) \frac{du}{dx}(0)$
$- \alpha\, v(0) u(0)$

$$F(v) = (v, \tan x) + \alpha v(0)$$

Exc 2 $a(v,u) = \left(\frac{dv}{dx}, \cos x \frac{du}{dx}\right) + \left(v, \frac{du}{dx}\right)$

$+ (1+\alpha)\, v(0) \frac{du}{dx}(0) - \alpha\, v(0) u(0)$

- $a(u,u) = \int_0^1 \frac{du}{dx} \cos x \frac{du}{dx}\, dx + \int_0^1 u \underbrace{\frac{du}{dx}}\, dx$ — $\frac{d}{dx}(u^2)$

$+ (1+\alpha)\, u(0) \frac{du}{dx}(0) - \alpha\, u(0) u(0)$

$a(u,u) = \int_0^1 \cos x \left(\frac{du}{dx}\right)^2 dx + \frac{1}{2} u^2 \Big|_0^1 + (1+\alpha) u(0) \frac{du}{dx}(0)$
$- \alpha u(0)^2$

0.5 $u(1)=0$

$a(u,u) \overset{\downarrow}{=} \underbrace{\int_0^1 \cos x \left(\frac{du}{dx}\right)^2 dx}_{\geq 0 \text{ on } [0,1]} \underbrace{- \left(\frac{1}{2}+\alpha\right) u(0)^2}_{\geq 0 \text{ if } -\left(\frac{1}{2}+\alpha\right) \geq 0} + \underbrace{(1+\alpha) u(0) \frac{du}{dx}(0)}_{\text{sign unknown}}$

sign unknown $\Rightarrow \alpha = -1$ to cancel term

for $\alpha = -1$ indeed $-\left(\frac{1}{2}+\alpha\right) = \frac{1}{2} \geq 0$

$\Rightarrow a(u,u) \geq 0$ if $\alpha = -1$ $\Rightarrow a(\cdot,\cdot)$ non-negative

- is $a(u,u)$ positive definite

$a(u,u) = 0 \Rightarrow \int_0^1 \cos x \left(\frac{du}{dx}\right)^2 dx + \frac{1}{2} u(0)^2 = 0$

0.5 only if $\frac{du}{dx} = 0 \Rightarrow u = c^{st}$ and $u(0) = 0$ } $u \in \mathcal{V}$, hence continuous $\Rightarrow u = 0$

hence $a(u,u) \geq 0$ and $a(u,u) = 0$ only if $u = 0$

$\Rightarrow a(\cdot,\cdot)$ positive definite

- is $a(\cdot,\cdot)$ coercive

$a(u,u) = \int_0^1 \cos x \left(\frac{du}{dx}\right)^2 dx + \frac{1}{2} u(0)^2 \geq \int_0^1 \cos x \left(\frac{du}{dx}\right)^2 dx$

0.5 $\geq \min_{x \in [0,1]} \cos x \int_0^1 \left(\frac{du}{dx}\right)^2 dx = \cos 1 \left\| \frac{du}{dx} \right\|^2$

$= \cos 1 \left(\frac{1}{2} \left\| \frac{du}{dx} \right\|^2 + \frac{1}{2} \left\| \frac{du}{dx} \right\|^2 \right) \underset{(*)}{\geq} \frac{\cos 1}{2} \left(\left\| \frac{du}{dx} \right\|^2 + \|u\|^2 \right)$

$= \frac{\cos 1}{2} \|u\|_{H^1}$

$\Rightarrow a(\cdot,\cdot)$ coercive

(*): Poincare inequality with $L=1$, using $u(1)=0$:
$\left\| \frac{du}{dx} \right\|^2 \geq \|u\|^2$

Exc 3 Lax-Milgram theorem

assume $\mathcal{V}$ Hilbert space, $a(v,u)$ bilinear form, $F(v)$ linear form

the problem: find $u \in \mathcal{V}$ s.t $a(v,u) = F(v) \quad \forall v \in \mathcal{V}$

has a unique solution if $\forall u, v \in \mathcal{V}$

0.6

1. $a(u,u) \geq c\|u\|_{\mathcal{V}}^2$ for some $c > 0$

2. $|a(v,u)| \leq M \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}$

3. $|F(v)| \leq K \|v\|_{\mathcal{V}}$

$\mathcal{V} = \{v \in H^1 \mid v(1) = 0\} \Rightarrow \|\cdot\|_{\mathcal{V}} = \|\cdot\|_{H^1}$

0.1 1. already proven in Exercise 2

0.6

2. $$|a(v,u)| = \left|\left(\frac{dv}{dx}, \cos x \frac{du}{dx}\right) + \left(v, \frac{du}{dx}\right) + v(0)u(0)\right|$$

$$\leq \left|\left(\frac{dv}{dx}, \cos x \frac{du}{dx}\right)\right| + \left|\left(v, \frac{du}{dx}\right)\right| + |v(0)u(0)|$$

$$\overset{C.S.}{\leq} \left\|\frac{dv}{dx}\right\| \left\|\cos x \frac{du}{dx}\right\| + \|v\| \left\|\frac{du}{dx}\right\| + |v(0)|\,|u(0)|$$

$$\overset{(*)}{\leq} \left\|\frac{dv}{dx}\right\| \left\|\frac{du}{dx}\right\| + \|v\| \left\|\frac{du}{dx}\right\| + \left\|\frac{dv}{dx}\right\| \left\|\frac{du}{dx}\right\|$$

$$\leq 3\sqrt{\left\|\frac{dv}{dx}\right\|^2 + \|v\|^2}\sqrt{\left\|\frac{du}{dx}\right\|^2 + \|u\|^2}$$

$$= 3\|v\|_{H^1} \|u\|_{H^1}$$

(*) i. $$\left\|\cos x \frac{du}{dx}\right\| \overset{C.S}{\leq} \left\|\frac{du}{dx}\right\| \|\cos(x)\| \leq \left\|\frac{du}{dx}\right\|$$

ii $$v(1) = v(0) + \int_0^1 \frac{dv}{dx}\,dx \overset{\text{since } v(1)=0}{\Rightarrow} v(0) = -\int_0^1 \frac{dv}{dx}\,dx$$

$$\Rightarrow |v(0)| \overset{C.S.}{\leq} \|1\| \left\|\frac{dv}{dx}\right\| = \left\|\frac{dv}{dx}\right\|$$

0.6

3. $$|F(v)| = |(v, \tan x) - v(0)| \leq |(v, \tan x)| + |v(0)|$$

$$\leq \|v\| \|\tan x\| + \left\|\frac{dv}{dx}\right\|$$

$$\leq \tan 1 \sqrt{\|v\|^2 + \left\|\frac{dv}{dx}\right\|^2} + \sqrt{\|v\|^2 + \left\|\frac{dv}{dx}\right\|^2}$$

$$= (\tan 1 + 1)\|v\|_{H^1}$$

0.1 $\Rightarrow$ weak form well posed

- associated minimization problem if

  $a(\cdot,\cdot)$ symmetric

  $a(\cdot,\cdot)$ pos. definite/coercive

still to show $a(\cdot,\cdot)$ symmetric or not

$$a(v,u) = \underbrace{\int_0^1 \frac{dv}{dx} \cos x \frac{du}{dx}\, dx}_{\text{symmetric}} + \int_0^1 v \frac{du}{dx}\, dx + \underbrace{v(0)\, u(0)}_{\text{symmetric}}$$

0.5 $$\left(v, \frac{du}{dx}\right) = \int_0^1 v \frac{du}{dx}\, dx = vu\Big|_0^1 - \int_0^1 \frac{dv}{dx} u\, dx$$

$$\underset{v(1)=0}{=} -v(0)\,u(0) - \int_0^1 \frac{dv}{dx} u\, dx = -\left(\frac{dv}{dx}, u\right) - v(0)u(0)$$

$\Rightarrow a(\cdot,\cdot)$ not symmetric

$\Rightarrow$ no an associated minimization problem

Exc 4 $\quad V_h = \text{span}\{\phi_0, \dots, \phi_{n-1}\}$ i.e $u = \sum_{i=0}^{n-1} c_i \phi_i(x)$

0.4
- matrix A s.t $A_{ij} = a(\phi_i, \phi_j)$
- from exc 2 : $a(u,u) > 0$ for $u \neq 0$

0.1 take $u = \sum_{i=0}^{n-1} c_i \phi_i \Rightarrow a\left(\sum_{i=0}^{n-1} c_i \phi_i, \sum_{j=0}^{n-1} c_j \phi_j\right) > 0$ for $\vec{c} \neq 0$

$(u \neq 0 \Leftrightarrow \vec{c} \neq 0)$

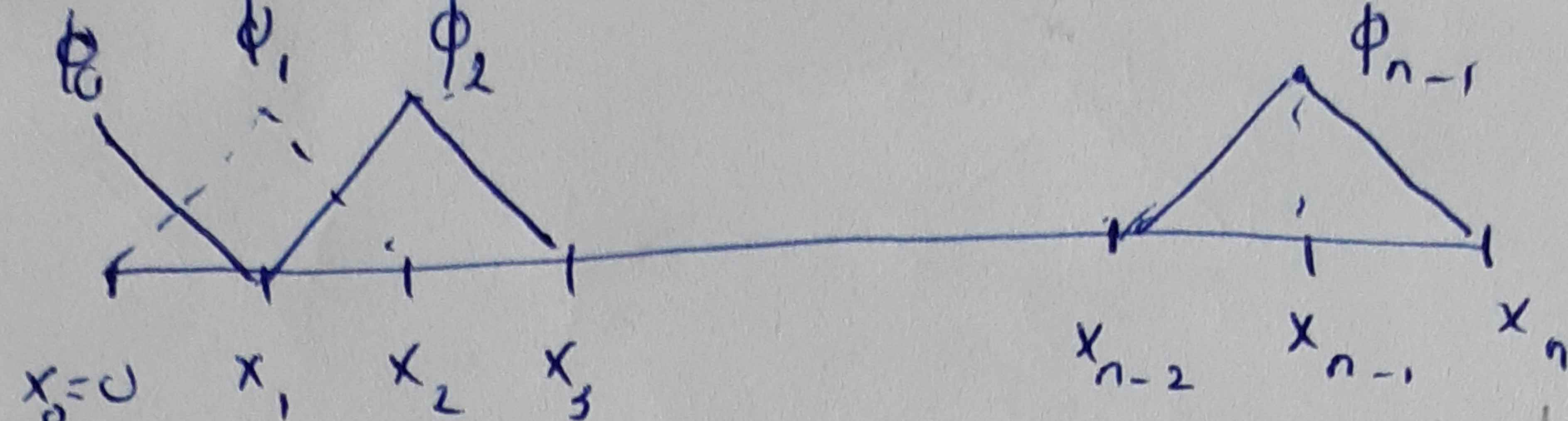
$$\Rightarrow \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} c_i\, a(\phi_i, \phi_j)\, c_j > 0 \quad \text{for } \vec{c} \neq 0$$

0.4 $$\Rightarrow \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} c_i A_{ij} c_j = \vec{c}^{\,T} A \vec{c} = (\vec{c}, A c) > 0$$

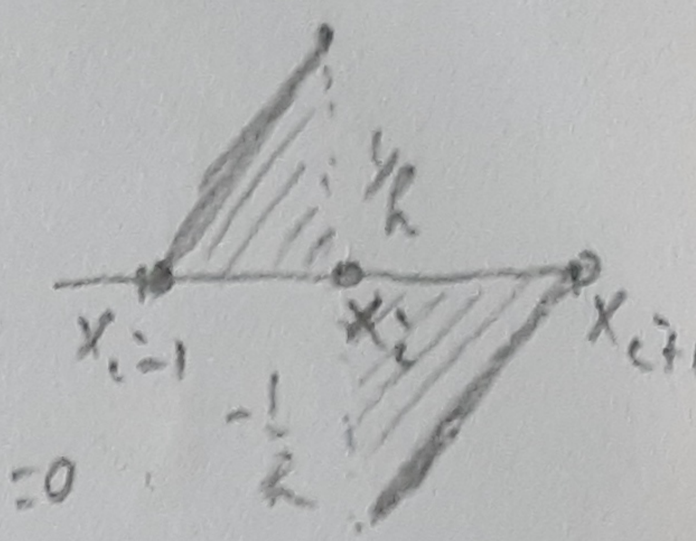
for $\vec{c} \neq 0$

0.1 $\Rightarrow A$ positive definite

Exc 5



$(*)$ $\phi_i \dfrac{d\phi_i}{dx}$:



$\Rightarrow \displaystyle\int_{x_{i-1}}^{x_{i+1}} \phi_i \frac{d\phi_i}{dx}\, dx = 0$

- $\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$\phi_i(x)$ piecewise linear, linear on $[x_j, x_{j+1}]$ $j = 0, \dots, n-1$

0.5 i.e $\phi_i(x) = \begin{cases} 0 & x \le x_{i-1} \text{ and } x \ge x_{i+1} \\ \dfrac{x - x_{i-1}}{h} & x_{i-1} < x \le x_i \\ \dfrac{x_{i+1} - x}{h} & x_i < x < x_{i+1} \end{cases}$

- hence $\dfrac{d\phi_i}{dx}(x) = \begin{cases} 0 & x \le x_{i-1} \text{ and } x \ge x_{i+1} \\ 1/h & x_{i-1} < x \le x_i \\ -\dfrac{1}{h} & x_i < x < x_{i+1} \end{cases}$

0.3

- $A_{ii} = a(\phi_i, \phi_i) = \left(\dfrac{d\phi_i}{dx}, \cos x \dfrac{d\phi_i}{dx}\right) + \left(\phi_i, \dfrac{d\phi_i}{dx}\right) + (\phi_i(0))^2$

$i=0$: $\phi_0(0) = 1$

$$A_{00} = \int_0^h \left(-\frac{1}{h} \cdot -\frac{1}{h} \cos x\right) dx + \int_0^h \frac{h-x}{h} \cdot -\frac{1}{h}\, dx + 1^2$$

0.6 $$= \frac{1}{h^2} \sin x \Big|_0^h - \frac{1}{h^2}\left(-\frac{1}{2}x^2 + hx\right)\Big|_0^h + 1^2$$

$$= \frac{\sin h}{h^2} - \frac{1}{h^2} \cdot \frac{1}{2} h^2 + 1 = \frac{\sin h}{h^2} + \frac{1}{2}$$

$i = 1, \dots, n-1$: $\phi_i(0) = 0$

$$A_{ii} = \frac{1}{h^2} \int_{x_{i-1}}^{x_{i+1}} \cos x\, dx + \underbrace{\int_{x_{i-1}}^{x_i} \frac{1}{h} \frac{(x - x_{i-1})}{h} dx + \int_{x_i}^{x_{i+1}} -\frac{1}{h} \frac{(x_{i+1} - x)}{h} dx}_{=0\ (*)}$$

0.6 $$= \frac{1}{h^2} \sin x \Big|_{(i-1)h}^{(i+1)h} = \frac{1}{h^2}(\sin(ih + h) - \sin(ih - h))$$

$$= \frac{2 \cos ih \sin h}{h^2}$$